Resit Exam Percolation Theory

2 February 2023, 18:15-20:15

- It is absolutely not allowed to use calculators, phones, computers, books, notes, the help of others or any other aids.
- Write the answer to each question on a separate sheet, **with your name and student number on each sheet**. This is worth 10 points (out of a total of 100).

Exercise 1 (20 pts).
State and prove the BK inequality.

Exercise 2 (20 pts)
Show that

$$p_c(\mathbb{Z}^d) \leq p_c(\mathbb{Z}^d, \text{site}) \leq 1 - \left(1 - p_c(\mathbb{Z}^d)\right)^{2d},$$

where $p_c(\mathbb{Z}^d, \text{site})$ refers to the *site percolation model* where vertices (sites) are open with probability p and closed with probability $1-p$; and we say percolation occurs if there exists an infinite path all of whose vertices are open. (Each edge of $\mathbb{Z}^d$ is automatically open.)

Exercise 3 (20 pts).
There is an election between two candidates ($+1$ and -1), and each of the n voters is equally likely to choose either candidate and independently of all other voters. (This is called the "impartial culture assumption" in social choice theory.) The outcome of the election is given by a function $f : \{\pm 1\}^n \to \{\pm 1\}$, which is non-decreasing in each coordinate. Suppose that the $(+1)$-candidate, who happens to be very wealthy, has a (small) chance $\mathbb{P}(f = +1) \geq .01$ to win the election.

Show that there exists a universal constant c (independent of n and the precise choice of f), such that by bribing no more than $c \cdot \frac{n}{\ln n}$ voters to set their vote to $+1$, the $(+1)$-candidate can increase his chance of winning to .99.

More formally formulated : Show that there exists a constant, such that for all n and every $f : \{\pm 1\}^n \to \{\pm 1\}$ that is non-decreasing in all coordinates and satisfies

$$\mathbb{P}(f(X_1, \ldots, X_n) = 1) \geq .01,$$

where $X_1, \ldots, X_n$ are i.i.d. uniform on $\{\pm 1\}$, there exists a set $I \subseteq [n]$ with $|I| \leq c \cdot \frac{n}{\ln n}$ such that

$$\mathbb{P}(f(Y_1, \ldots, Y_n) = 1) \geq .99,$$

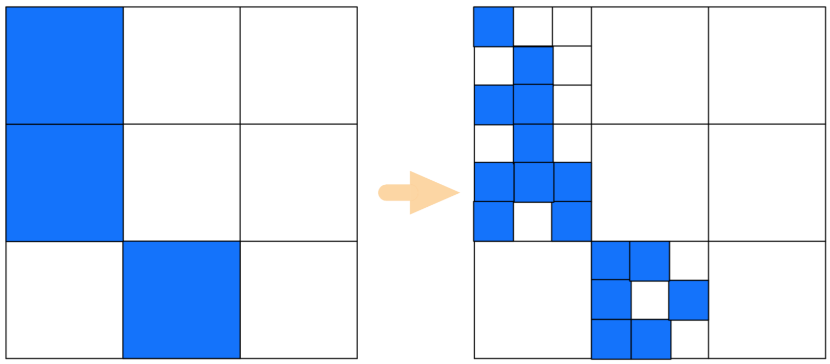
where

$$Y_j := \begin{cases} 1 & \text{if } j \in I, \\ X_j & \text{otherwise.} \end{cases}$$

(*Hint*: Talagrand and/or KKL.)

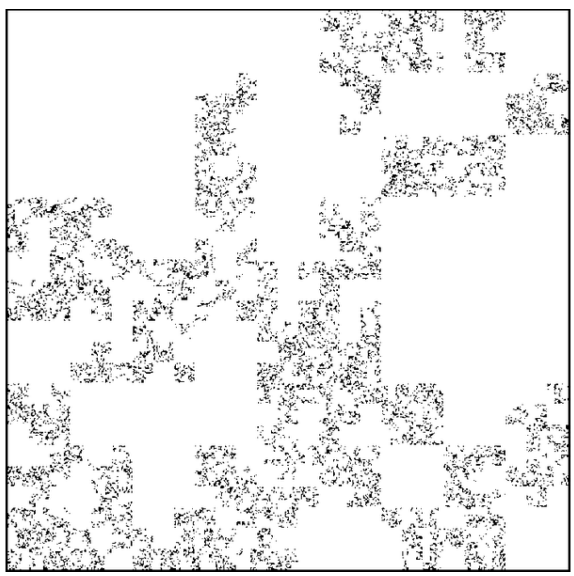
(See next page)

Exercise 4 (a:5,b:5,c:4,d:4,e:4,f:4,g:4 pts)
Fractal percolation refers to the following process. We dissect the plane into axis parallel squares with side length one in the obvious, standard way. Each of these we further dissect into $9 = 3 \times 3$ axis parallel squares of side length $1/3$, each of which we keep with probability p and discard with probability $1 - p$. We dissect each surviving side-length $1/3$ square into 9 squares of side length $1/9$, which we keep with probability p and discard with probability $1 - p$.



(Depiction of the first two steps of the contruction.)

We keep repating this process ad infinitum, and let A be the (possibly empty) set of all points in $\mathbb{R}^2$ that have never been removed.



(Computer simulation of the set A.)

We can define the process more formally as follows. For all $i, j \in \mathbb{Z}, n \in \{0\} \cup \mathbb{N}$, we let

$$S_{i,j,n} := \left[\frac{i}{3^n}, \frac{i+1}{3^n}\right] \times \left[\frac{j}{3^n}, \frac{j+1}{3^n}\right],$$

and we let $X_{i,j,n}$ be independent Bernoulli random variables with parameter p.

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For $n \geq 1$ we now set

$$B_n := \bigcup_{\substack{i,j\in\mathbb{Z},\\ X_{i,j,n}=1}} S_{i,j,n}, \quad A_n := \bigcap_{m=1}^{n} B_m, \quad A := \bigcap_{n\geq 1} B_n.$$

a) Show that

$$\mathbb{P}_p(A = \emptyset) = \begin{cases} 1 & \text{if } p \leq 1/9, \\ 0 & \text{if } p > 1/9. \end{cases}$$

(*Hint:* you may wish to use a coupling with a suitable Galton-Watson branching process. Recall from lectures that a Galton-Watson branching process with offspring distribution X starts with a single individual that has a random number of "children" – following the distribution of X – and each descendant again has a random number of children distributed like X and independent of the numbers of children of other individuals in the same or previous generations. You may use without proof the "main theorem of branching processes", stating that – provided $0 < \mathbb{P}(X = 0) < 1$ – the process dies out almost surely if and only if $\mathbb{E}X \leq 1$.)

b) Let D be the event that A is "dust", i.e. for every distinct $a, b \in A$ there is no continuous curve between a and b that is completely contained in A. Show that $\mathbb{P}_p(D) = 1$ if $p \leq 1/\sqrt{3}$.

(*Hint:* as a first step, you could consider a line segment I of the form $[0, 1] \times \{\frac{i}{3^n}\}$ with $i \in \mathbb{Z}$, and show that almost surely A does not contain any curve that crosses this line segment. Again you may wish to use a comparison to a suitable branching process. If at least one of the squares of side length one that have I as one of their sides is discarded, what can you say about the existence of curves crossing I?)

As you might expect, we'll say that *percolation* occurs if A contains an unbouded (continuous) curve. In the remainder of the exercise we wish to show the existence of a critical probability $p_c < 1$ such that $\mathbb{P}_p(\text{percolation}) > 0$ for all $p > p_c$. (Side remark: by the previous part $p_c \geq 1/\sqrt{3}$.)

We'll say a square $S_{i,j,n}$ is 1-good if it if at least 8 of 9 the side length $1/3^{n+1}$ squares $S_{3i,3j,n+1}, \ldots, S_{3i+2,3j+2,n+1}$ (whose union is $S_{i,j,n}$) are not removed. Phrased differently, $S_{i,j,n}$ is 1-good if $X_{3i,3j,n+1} + \cdots + X_{3i+2,3j+2,n+1} \geq 8$.

For $m \geq 2$ we'll say $S_{i,j,n}$ is m-good if at least 8 of the side length $1/3^{n+1}$ squares whose union is $S_{i,j,n}$ are $(m-1)$-good and are not removed. A square is ∞-good if it is m-good for all $m \geq 1$.

In the current setting, we'll say a *horizontal crossing* of the axis-parallel rectangle $R = [a, b] \times [c, d]$ is a continuous curve contained entirely in R, starting from a point on the left boundary $\{a\} \times [c, d]$ and ending in a point of the right boundary $\{b\} \times [c, d]$. A vertical crossing is defined analogously.

c) Explain why if the unit side length square $S_{0,0,0} = [0, 1]^2$ is m-good then A_m contains both a horizontal and a vertical clossing of $S_{0,0,0}$; and if $S_{0,0,0}$ and $S_{1,0,0}$ are both m-good then A_m contains a horizontal crossing of $S_{0,0,0} \cup S_{1,0,0} = [0, 2] \times [0, 1]$.

(A few well-chosen sentences will suffice here.)

Using a standard compactness argument (which we spare you at the present occasion), it also follows that if $S_{0,0,0}$ is ∞-good then A contains both a horizontal and vertical crossing of $S_{0,0,0}$ and if $S_{0,0,0}$ and $S_{1,0,0}$ are both ∞-good then A contains a horizontal crossing of $S_{0,0,0} \cup S_{1,0,0}$.

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We denote

$$\vartheta_m := \mathbb{P}_p(S_{0,0,0} \text{ is } m\text{-good}), \quad \vartheta := \mathbb{P}_p(S_{0,0,0} \text{ is } \infty\text{-good}).$$

d) Explain why

$$\vartheta_1 = \varphi(1), \quad \vartheta_{m+1} = \varphi(\vartheta_m) \quad (\forall m \geq 0),$$

where

$$\varphi(x) = p^9 \cdot \big(x^9 + 9x^8(1-x)\big) + 9p^8(1-p)x^8.$$

e) Show that ϑ is the largest solution in $[0,1]$ of the equation

$$x = \varphi(x).$$

f) Show that for every $0 \leq \alpha < 1$ there is a $p_0 = p_0(\alpha) < 1$ such that $\vartheta > \alpha$ provided we chose the parameter $p \geq p_0$.

(*Hint:* you could try to show that $\varphi(x) - x$ will change sign on $(\alpha, 1)$ provided p is chosen sufficiently close to 1.)

g) Explain how it follows that $p_c < 1$.

(The end)